

The Caffarelli-Kohn-Nirenberg inequalities and manifolds with nonnegative weighted Ricci curvature

Jing Mao

Department of Mathematics, Harbin Institute of Technology (Weihai), Weihai, 264209, China
Email: jiner120@163.com, jiner120@tom.com

Abstract

We prove that n -dimensional ($n \geq 3$) complete and non-compact metric measure spaces with non-negative weighted Ricci curvature in which some Caffarelli-Kohn-Nirenberg type inequality holds are *close* to the model metric measure n -space (i.e., the Euclidean metric n -space).

1 Introduction

Denote by $C_0^\infty(\mathbb{R}^n)$ the space of smooth functions with compact support in the n -dimensional Euclidean space. Let $n \geq 3$ be an integer and let a, b, p be constants satisfying the following conditions

$$-\infty < a < \frac{n-2}{2}, \quad a \leq b \leq a+1, \quad p = \frac{2n}{n-2+2(b-a)}. \quad (1.1)$$

For all $u \in C_0^\infty(\mathbb{R}^n)$, Caffarelli, Kohn and Nirenberg [6] has proven that there exists a positive constant C depending only on constants a, b and n (these constants satisfy (1.1) above) such that the functional equality

$$\left(\int_{\mathbb{R}^n} |x|^{-bp} |u|^p dv_{\mathbb{R}^n} \right)^{\frac{1}{p}} \leq C \left(\int_{\mathbb{R}^n} |x|^{-2a} |\nabla u| dv_{\mathbb{R}^n} \right)^{\frac{1}{2}} \quad (1.2)$$

holds, where $|x|$ is the Euclidean length of $x \in \mathbb{R}^n$, and $dv_{\mathbb{R}^n}$ is the Euclidean volume element determined by the standard Euclidean metric. We know that when $a = b = 0$, the Caffarelli-Kohn-Nirenberg type inequality (1.2) degenerates into the classical Sobolev inequality; when $a = 0$ and $b = 1$, the Caffarelli-Kohn-Nirenberg type inequality (1.2) becomes the Hardy inequality. The Sobolev and the Hardy inequalities have many important applications (see, e.g., [2, 3, 6, 8, 9, 14, 15, 18, 19, 24, 25] and the references therein), so it is meaningful to investigate the Caffarelli-Kohn-Nirenberg type inequality (1.2). The sharpest constant C such that the inequality (1.2) holds

MSC 2010: 53C21; 31C12.

Key Words: Caffarelli-Kohn-Nirenberg type inequality; Volume comparison; Weighted Ricci curvature.

is called the *best* constant. In the study of functional inequalities, finding the best constants is also interesting and difficult subject.

Let $K_{a,b}$ be the best constant for the Caffarelli-Kohn-Nirenberg type inequality (1.2), which implies

$$K_{a,b}^{-1} = \inf_{u \in C_0^\infty(\mathbb{R}^n) - \{0\}} \frac{\left(\int_{\mathbb{R}^n} |x|^{-2a} |\nabla u|^2 dv_{\mathbb{R}^n} \right)^{\frac{1}{2}}}{\left(\int_{\mathbb{R}^n} |x|^{-bp} |u|^p dv_{\mathbb{R}^n} \right)^{\frac{1}{p}}}. \quad (1.3)$$

There exist several conclusions related to the best constant $K_{a,b}$. More precisely, for the Sobolev inequality (corresponding to the case of $a = b = 0$), Aubin [1] and Talenti [30] have *separately* shown that

$$K_{0,0} = \left(\frac{1}{n(n-2)} \right)^{\frac{1}{2}} \left(\frac{\Gamma(n)}{nD_n\Gamma^2\left(\frac{n}{2}\right)} \right)^{\frac{1}{n}},$$

where D_n is the volume of the unit ball in \mathbb{R}^n , and that a family of minimizers is given by

$$u(x) = (\lambda + |x|^2)^{1-\frac{n}{2}}, \quad \lambda > 0.$$

For the case of $a = 0$ and $0 < b < 1$, Lieb [19] has proven that the best constant $K_{0,b}$ is

$$K_{0,b} = \left(\frac{1}{(n-2)(n-bp)} \right)^{\frac{1}{2}} \left(\frac{(2-bp)\Gamma\left(\frac{2(n-bp)}{2-bp}\right)}{nD_n\Gamma^2\left(\frac{n-bp}{2-bp}\right)} \right)^{\frac{2(n-bp)}{2-bp}},$$

and that a family of minimizers is given by

$$u(x) = (\lambda + |x|^{2-bp})^{-\frac{n-2}{2-bp}}, \quad \lambda > 0.$$

Chou and Chu [11] have improved the above two cases to the situation that $a \geq 0$, $a \leq b < a + 1$, and have shown that the best constant $K_{a,b}$ is

$$K_{a,b} = \left(\frac{1}{(n-2a-2)(n-bp)} \right)^{\frac{1}{2}} \left(\frac{(2-bp+2a)\Gamma\left(\frac{2(n-bp)}{2-bp+2a}\right)}{nD_n\Gamma^2\left(\frac{n-bp}{2-bp+2a}\right)} \right)^{\frac{2(n-bp)}{2-bp+2a}},$$

and that, for $a > 1$, all minimizers are non-zero constant multiples of the function

$$u(x) = (\lambda + |x|^{2-bp+2a})^{-\frac{n-2-2a}{2-bp+2a}}, \quad \lambda > 0.$$

Catrina and Wang [9] have investigated the remaining case of the best constant $K_{a,b}$ and the existence or non-existence of the minimizers.

From now on, we fix some notations, that is, let $n \geq 3$ be an integer, and let a , b and p be constants satisfying

$$0 \leq a < \frac{n-2}{2}, \quad a \leq b < a + 1, \quad p = \frac{2n}{n-2+2(b-a)}. \quad (1.4)$$

For a prescribed complete manifold, denote by $C_0^\infty(M)$ the space of smooth functions with compact support on M , and let dv_g be the volume element (i.e., Riemannian measure) related to the Riemannian metric g . In this paper, for convenience, we make an agreement that $\text{vol}(\cdot)$ represents the volume of the given geometric object.

Given a complete open manifold M with non-negative Ricci curvature, do Carmo and Xia [8] have revealed the following potential relation between a Caffarelli-Kohn-Nirenberg type inequality (on M) of the form (1.2) and the geometric property related to the volume of a geodesic ball on M .

Theorem 1.1. ([8]) *Let $C_1 \geq K_{a,b}$ be a constant, with $K_{a,b}$ determined by (1.3) and (1.4), and M be an n -dimensional ($n \geq 3$) complete open manifold with non-negative Ricci curvature. Fix a point $x_0 \in M$ and denote by ρ the distance function on M from x_0 . Assume that, for any $u \in C_0^\infty(M)$, we have*

$$\left(\int_M t^{-bp} |u|^p dv_g \right)^{\frac{1}{p}} \leq C_1 \left(\int_M t^{-2a} |\nabla u|^2 dv_g \right)^{\frac{1}{2}}.$$

Then for any $x \in M$, we have

$$\text{vol}[B(x_0, r)] \geq \left(\frac{K_{a,b}}{C_1} \right)^{\frac{n}{1+a-b}} \cdot V_0(r), \quad \forall r > 0,$$

where $V_0(r)$ is the volume of an r -ball in \mathbb{R}^n .

For the special case that $a = b = 0$, the above theorem is covered by [32, Theorem 2]. We prefer to point out one thing here, that is, [32, Theorem 2] has been improved by Mao [24] recently (see [24, Theorem 1.3] for the precise statement or the end of Section 1 of [25] for the detailed explanation).

The purpose of this paper is to generalize Theorem 1.1 above. For that, we need to use the following notions of smooth metric measure spaces and the weighted Ricci curvature.

A smooth metric measure space (also known as the weighted measure space) is actually a Riemannian manifold equipped with some measure which is conformal to the usual Riemannian measure. More precisely, for a given complete n -dimensional Riemannian manifold (M, g) with the metric g , the triple $(M, g, e^{-f} dv_g)$ is called a smooth metric measure space, where f is a *smooth real-valued* function on M and, as before, dv_g is the Riemannian volume element related to g (sometimes, we also call dv_g the volume density). Correspondingly, for a geodesic ball $B(x_0, r)$ on M , with center $x_0 \in M$ and radius r , one can also define its *weighted (or f -)volume* $\text{vol}_f[B(x_0, r)]$ as follows

$$\text{vol}_f[B(x_0, r)] := \int_{B(x_0, r)} e^{-f} dv_g.$$

Now, for convenience, we also make an agreement that in this paper $\text{vol}_f(\cdot)$ represents the weighted (or f -)volume of the given geometric object on a metric measure space.

For a given smooth metric measure space $(M, g, e^{-f} dv_g)$, the following N -Bakry-Émery tensor

$$\text{Ric}_f^N := \text{Ric} + \text{Hess}f - \frac{df \otimes df}{N},$$

with Ric and Hess the Ricci and the Hessian operators on M , can be considered. Especially, when $N = \infty$, the N -Bakry-Émery tensor Ric_f^N degenerates into the so-called ∞ -Bakry-Émery Ricci tensor Ric_f which is given by

$$\text{Ric}_f = \text{Ric} + \text{Hess}f.$$

The ∞ -Bakry-Émery Ricci tensor is also called *the weighted Ricci tensor*. Bakry and Émery [4, 5] introduced firstly and extensively investigated the generalized Ricci tensor above and its relationship with diffusion processes.

Similar to the p -norm of smooth functions with compact support on the manifold (M, g) , for the smooth metric measure space $(M, g, e^{-f} dv_g)$ and any $u \in C_0^\infty(M)$, we can define *the weighted p -norm* $\|u\|_{p;MMS}$ of u as follows

$$\|u\|_{p;MMS} := \left(\int_M |u|^p \cdot e^{-f} dv_g \right)^{\frac{1}{p}}.$$

Clearly, when $f \equiv 0$, the weighted p -norm is just the p -norm.

Maybe people would have an illusion that smooth metric measure spaces are not necessary to study since they are simply obtained from correspondingly Riemannian manifolds by adding a conformal measure to the Riemannian measure. However, the truth is not like this, and they do have many differences. For instance, when Ric_f is bounded from below, the Myer's theorem, Bishop-Gromov's volume comparison, Cheeger-Gromoll's splitting theorem and Abresch-Gromoll excess estimate cannot hold as the Riemannian case. Here, for the purpose of comprehension, we would like to repeat an example given in [31, Example 2.1]. That is, for the metric measure space $(\mathbb{R}^n, g_{\mathbb{R}^n}, e^{-f} dv_{g_{\mathbb{R}^n}})$, where $g_{\mathbb{R}^n}$ is the usual Euclidean metric and $dv_{g_{\mathbb{R}^n}}$, as before, is the Euclidean volume density related to $g_{\mathbb{R}^n}$, if $f(x) = \frac{\lambda}{2}|x|^2$ for $x \in \mathbb{R}^n$, then we have $\text{Hess} = \lambda g_{\mathbb{R}^n}$ and $\text{Ric}_f = \lambda g_{\mathbb{R}^n}$. Therefore, from this example, we know that unlike in the case of Ricci curvature bounded from below uniformly by some positive constant, a metric measure space is not necessarily compact provided $\text{Ric}_f \geq \lambda$ and $\lambda > 0$. So, it is meaningful to study the geometry of smooth metric measure spaces. For the basic and necessary knowledge about the metric measure spaces, we refer readers to the excellent work [31] of Wei and Wylie. The subject on the metric measure space and the related weighted Ricci tensor occurs naturally in many different subjects and has many important applications (see, e.g., [20, 26, 31]).

Theorem 1.2. *Let $C_2 \geq K_{a,b}$ be a constant, where $K_{a,b}$ is determined by (1.3) and (1.4). Assume that $(M, g, e^{-f} dv_g)$ is an n -dimensional ($n \geq 3$) complete and noncompact smooth metric measure space with non-negative weighted Ricci curvature. For a point $x_0 \in M$ at which $f(x_0)$ is away from $-\infty$, assume that the radial derivative $\partial_t f$ satisfies $\partial_t f \geq 0$ along all minimal geodesic segments from x_0 , with $t := d(x_0, \cdot)$ the distance to x_0 (on M). If furthermore for any $u \in C_0^\infty(M)$, the Caffarelli-Kohn-Nirenberg type inequality*

$$\left(\int_M t^{-bp} |u|^p \cdot e^{-f} dv_g \right)^{\frac{1}{p}} \leq C_2 \left(\int_M t^{-2a} |\nabla u|^2 \cdot e^{-f} dv_g \right)^{\frac{1}{2}} \quad (1.5)$$

holds, then we have

$$\text{vol}_f[B(x_0, r)] \geq \left(\frac{K_{a,b}}{C_2} \right)^{\frac{n}{1+a-b}} \cdot e^{-f(x_0)} \cdot V_0(r), \quad \forall r > 0,$$

where $V_0(r)$ is the volume of an r -ball in \mathbb{R}^n .

Remark 1.3. If $f \equiv 0$ on M , then the metric measure space $(M, g, e^{-f} dv_g)$ can be seen as the Riemannian manifold (M, g) directly. Clearly, in this case, Theorem 1.2 is totally the same with Theorem 1.1 above. So, we can equivalently say that [8, Theorem 1.1] is only a special case of Theorem 1.2. Moreover, as pointed out in [24, Remark 1.4] or [25, Remark 1.4], since f is a smooth real-valued function on the complete non-compact manifold M , we know that if $f(x)$ does not tend to $-\infty$ as x tends to the infinity, then x_0 can be chosen arbitrarily; if $f(x) \rightarrow -\infty$ as x tends to the infinity, then x_0 can be chosen to any point except those points near the infinity. Besides, we say that Theorem 1.2 here is *sharper* than Theorem 1.1, since at x_0 , one can always set up a global polar coordinate chart $\{t, \xi\}$ with $(t, \xi) \in [0, +\infty) \times \mathbb{S}^{n-1}$ for the complete non-compact manifold M , where $t := d(x_0, \cdot)$ as in Theorem 1.2, and then we have $e^{-f(t, \xi)} \leq e^{-f(0, \xi)} = e^{-f(x_0)}$ by applying the assumption $\partial_t f \geq 0$ along all minimal geodesic segments from x_0 , which leads to

$$\text{vol}[B(x_0, r)] \geq e^{f(x_0)} \cdot \text{vol}_f[B(x_0, r)] = e^{f(x_0)} \int_{B(x_0, r)} e^{-f} dv_g \geq \left(\frac{K_{a,b}}{C_2} \right)^{\frac{n}{1+a-b}} \cdot V_0(r), \quad \forall r > 0$$

by using the conclusion of Theorem 1.2 directly. The assumption on f (i.e., *finite* at x_0 and *monotone non-decreasing* in the radial direction) in Theorem 1.2 seems a little strong here. However, it is not difficult to see that there are many examples satisfying this condition. For instance, as mentioned in [24, Remark 1.4], in the polar coordinate chart $\{t, \xi\}$ constructed above, one can choose $f(t, \xi) = t$ for $t \geq 0$. Even more, one can find that functions $f(t, \xi) = t^\ell$, $\ell > 1$, for $t \geq 0$ are also acceptable. Hence, from this aspect, Theorem 1.2 generalizes Theorem 1.1 a lot.

By applying Theorem 1.2 above, [12, Theorem 3.3, Corollary 3.4 and Theorem 4.2] (see also Theorems 2.5 and 2.6 in Section 2), and [31, Theorem 1.2] (see also Theorem 2.7 in Section 2), we can prove the following rigidity theorem.

Corollary 1.4. Assume that $(M, g, e^{-f} dv_g)$ is an n -dimensional ($n \geq 3$) complete and noncompact smooth metric measure space with non-negative weighted Ricci curvature. For a point $x_0 \in M$ at which $f(x_0)$ is away from $-\infty$, assume that the radial derivative $\partial_t f$ satisfies $\partial_t f \geq 0$ along all minimal geodesic segments from x_0 , with $t := d(x_0, \cdot)$ the distance to x_0 (on M). If furthermore for any $u \in C_0^\infty(M)$, the Caffarelli-Kohn-Nirenberg type inequality

$$\left(\int_M t^{-bp} |u|^p \cdot e^{-f} dv_g \right)^{\frac{1}{p}} \leq K_{a,b} \left(\int_M t^{-2a} |\nabla u|^2 \cdot e^{-f} dv_g \right)^{\frac{1}{2}}$$

holds, where $K_{a,b}$ is determined by (1.3) and (1.4), then (M, g) is isometric to $(\mathbb{R}^n, g_{\mathbb{R}^n})$. Moreover, in this case, we have $f \equiv f(x_0)$ is a constant function with respect to the variable t , and $e^{-f} dv_g = e^{-f(x_0)} dv_{g_{\mathbb{R}^n}}$. Here, as before, $g_{\mathbb{R}^n}$ and $dv_{g_{\mathbb{R}^n}}$ are the usual Euclidean metric and the Euclidean volume density related to $g_{\mathbb{R}^n}$, respectively.

It is interesting to know *under what kind of conditions* a complete open n -manifold ($n \geq 2$) is isometric to \mathbb{R}^n or has finite topological type, which in essence has relation with the splittingness of the prescribed manifold. This is a classical topic in the global geometry and has been studied extensively (see, e.g., [7, 21, 27]).

2 Useful facts

We would like to review [31, Theorem 1.2], which is the *cornerstone* of the proof of Theorem 1.2 shown in the next section, and [12, Theorem 3.3, Corollary 3.4 and Theorem 4.2], which are necessary to prove Corollary 1.4. However, before that, some necessary preliminaries should be introduced first. In fact, one can find more detailed versions (cf. [12, Section 2], [22, Section 2] and [23, Section 2.1 of Chapter 2]) of the following preliminaries, but we still give a simple version here so that readers can understand [31, Theorem 1.2] and [12, Theorem 3.3, Corollary 3.4 and Theorem 4.2]) completely and clearly.

2.1 Preliminaries

Denote by \mathbb{S}^{n-1} the unit sphere in \mathbb{R}^n . Given an n -dimensional ($n \geq 2$) complete Riemannian manifold (M, g) with the metric g , for a point $x \in M$, let S_x^{n-1} be the unit sphere with center x in the tangent space $T_x M$, and let $Cut(x)$ be the cut-locus of x , which is a closed set of zero n -Hausdorff measure. Clearly,

$$\mathbb{D}_x = \{t\xi \mid 0 \leq t < d_\xi, \xi \in S_x^{n-1}\}$$

is a star-shaped open set of $T_x M$, and through which the exponential map $\exp_x : \mathbb{D}_x \rightarrow M \setminus Cut(x)$ gives a diffeomorphism from \mathbb{D}_x to the open set $M \setminus Cut(x)$, where d_ξ is defined by

$$d_\xi = d_\xi(x) := \sup\{t > 0 \mid \gamma_\xi(s) := \exp_x(s\xi) \text{ is the unique minimal geodesic joining } x \text{ and } \gamma_\xi(t)\}.$$

As in [10], we can introduce two important maps used to construct the geodesic spherical coordinate chart at a prescribed point on a Riemannian manifold. For a fixed vector $\xi \in T_x M$, $|\xi| = 1$, let ξ^\perp be the orthogonal complement of $\{\mathbb{R}\xi\}$ in $T_x M$, and let $\tau_t : T_x M \rightarrow T_{\exp_x(t\xi)} M$ be the parallel translation along $\gamma_\xi(t)$. The path of linear transformations $\mathbb{A}(t, \xi) : \xi^\perp \rightarrow \xi^\perp$ is defined by

$$\mathbb{A}(t, \xi)\eta = (\tau_t)^{-1}Y_\eta(t),$$

where $Y_\eta(t) = d(\exp_x)_{(t\xi)}(t\eta)$ is the Jacobi field along $\gamma_\xi(t)$ satisfying $Y_\eta(0) = 0$, and $(\nabla_t Y_\eta)(0) = \eta$. Moreover, for $\eta \in \xi^\perp$, set $\mathcal{R}(t)\eta = (\tau_t)^{-1}R(\gamma'_\xi(t), \tau_t\eta)\gamma'_\xi(t)$, where the curvature tensor $R(X, Y)Z$ is defined by $R(X, Y)Z = -[\nabla_X, \nabla_Y]Z + \nabla_{[X, Y]}Z$. Then $\mathcal{R}(t)$ is a self-adjoint operator on ξ^\perp , whose trace is the radial Ricci tensor $\text{Ric}_{\gamma_\xi(t)}(\gamma'_\xi(t), \gamma'_\xi(t))$. Clearly, the map $\mathbb{A}(t, \xi)$ satisfies the Jacobi equation $\mathbb{A}'' + \mathcal{R}\mathbb{A} = 0$ with initial conditions $\mathbb{A}(0, \xi) = 0$, $\mathbb{A}'(0, \xi) = I$. By Gauss's lemma, the Riemannian metric of $M \setminus Cut(x)$ in the geodesic spherical coordinate chart can be expressed by

$$ds^2(\exp_x(t\xi)) = dt^2 + |\mathbb{A}(t, \xi)d\xi|^2, \quad \forall t\xi \in \mathbb{D}_x. \quad (2.1)$$

We consider the metric components $g_{ij}(t, \xi)$, $i, j \geq 1$, in a coordinate system $\{t, \xi_a\}$ formed by fixing an orthonormal basis $\{\eta_a, a \geq 2\}$ of $\xi^\perp = T_\xi S_x^{n-1}$, and then extending it to a local frame $\{\xi_a, a \geq 2\}$ of S_x^{n-1} . Define a function $J > 0$ on $\mathbb{D}_x \setminus \{x\}$ by

$$J^{n-1} = \sqrt{|g|} := \sqrt{\det[g_{ij}]}. \quad (2.2)$$

Since $\tau_t : S_x^{n-1} \rightarrow S_{\gamma_\xi(t)}^{n-1}$ is an isometry, we have

$$\langle d(\exp_x)_{t\xi}(t\eta_a), d(\exp_x)_{t\xi}(t\eta_b) \rangle_g = \langle \mathbb{A}(t, \xi)(\eta_a), \mathbb{A}(t, \xi)(\eta_b) \rangle_g,$$

and then $\sqrt{|g|} = \det \mathbb{A}(t, \xi)$. So, by applying (2.1) and (2.2), the volume $\text{vol}(B(x, r))$ of a geodesic ball $B(x, r)$, with radius r and center x , on M is given by

$$\text{vol}(B(x, r)) = \int_{S_x^{n-1}} \int_0^{\min\{r, d_\xi\}} \sqrt{|g|} dt d\sigma = \int_{S_x^{n-1}} \left(\int_0^{\min\{r, d_\xi\}} \det(\mathbb{A}(t, \xi)) dt \right) d\sigma, \quad (2.3)$$

where $d\sigma$ denotes the $(n-1)$ -dimensional volume element on $S^{n-1} \equiv S_x^{n-1} \subseteq T_x M$. As in Section 1, let $r(z) = d(x, z)$ be the intrinsic distance to the point $x \in M$. Since for any $\xi \in S_x^{n-1}$ and $t_0 > 0$, we have $\nabla r(\gamma_\xi(t_0)) = \gamma'_\xi(t_0)$ when the point $\gamma_\xi(t_0) = \exp_x(t_0\xi)$ is away from the cut locus of x (cf. [13]), then, by the definition of a non-zero tangent vector “*radial*” to a prescribed point on a manifold given in the first page of [16], we know that for $z \in M \setminus (\text{Cut}(x) \cup x)$ the unit vector field

$$v_z := \nabla r(z)$$

is the radial unit tangent vector at z . Set

$$l(x) := \max_{z \in M} r(z) = \max_{z \in M} d(x, z). \quad (2.4)$$

Then we have $l(x) = \max_\xi d_\xi$ (cf. [12, Section 2]). We also need the following fact about $r(z)$ (cf. Prop. 39 on p. 266 of [27]),

$$\partial_r \Delta r + \frac{(\Delta r)^2}{n-1} \leq \partial_r \Delta r + |\text{Hess}r|^2 = -\text{Ric}(\partial_r, \partial_r), \quad \text{with } \Delta r = \partial_r \ln(\sqrt{|g|}),$$

with $\partial_r = \nabla r$ as a differentiable vector (cf. Prop. 7 on p. 47 of [27] for the differentiation of ∂_r), where Δ is the Laplace operator on M and $\text{Hess}r$ is the Hessian of $r(z)$. Then, together with (2.2), we have

$$J'' + \frac{1}{(n-1)} \text{Ric}(\gamma'_\xi(t), \gamma'_\xi(t)) J \leq 0, \quad (2.5)$$

$$J(t, \xi) = t + O(t^2), \quad J'(t, \xi) = 1 + O(t). \quad (2.6)$$

As shown in [12] and also pointed out in [22], the facts (2.5) and (2.6) make a fundamental role in the derivation of *the generalized Bishop's volume comparison theorem I* below (see Theorem 2.5 for the precise statement). One can also find that (2.6) is also necessary in the proof of Theorem 1.2 in Section 3.

Denote by $\text{inj}(x)$ the injectivity radius of a point $x \in M$. Now, we would like to introduce a notion of spherically symmetric manifold which actually acts as the model space in this paper.

Definition 2.1. A domain $\Omega = \exp_x([0, l) \times S_x^{n-1}) \subset M \setminus \text{Cut}(x)$, with $l < \text{inj}(x)$, is said to be *spherically symmetric with respect to a point $x \in \Omega$* , if and only if the matrix $\mathbb{A}(t, \xi)$ satisfies $\mathbb{A}(t, \xi) = h(t)I$, for a function $h \in C^2([0, l))$, with $h(0) = 0$, $h'(0) = 1$, and $h|_{(0, l)} > 0$.

Naturally, Ω in Definition 2.1 is a spherically symmetric manifold and x is called its *base point*. Together with (2.1), on the set Ω given in Definition 2.1 the Riemannian metric of M can be expressed by

$$ds^2(\exp_x(t\xi)) = dt^2 + h^2(t)|d\xi|^2, \quad \xi \in S_x^{n-1}, \quad 0 \leq t < l, \quad (2.7)$$

with $|d\xi|^2$ the round metric on S^{n-1} . Spherically symmetric manifolds were named as *generalized space forms* by Katz and Kondo [16], and a standard model for such manifolds is given by the warped product $[0, l) \times_h S^{n-1}$ equipped with the metric (2.7), where h is called *the warping function* and satisfies the conditions of Definition 2.3.

For a spherically symmetric manifold $M^* := [0, l) \times_h S^{n-1}$ (with the base point p^*) and $r < l$, by (2.3) we have

$$\text{vol}[\tilde{B}(p^*, r)] = w_n \int_0^r h^{n-1}(t) dt, \quad (2.8)$$

and moreover, by the co-area formula (see, for instance, [10, pp. 85-86]), we also know that the volume of the boundary $\partial\tilde{B}(p^*, r)$ is given by $\text{vol}[\partial\tilde{B}(p^*, r)] = w_n h^{n-1}(r)$, where w_n denotes the $(n-1)$ -volume of the unit sphere in \mathbb{R}^n .

For more information about the spherically symmetric manifold $M^* = [0, l) \times_h S^{n-1}$ (e.g., the regularity of the metric of M^* , the asymptotically spectral properties the first Dirichlet eigenvalues of the Laplace and p -Laplace operators on M^* , etc.), please see [12, Section 2] and [22, Section 2] in detail.

2.2 Volume comparison theorems for manifolds with radial curvature bounded

As before, for the given complete manifold M , let $d(x, \cdot)$ be the Riemannian distance to x (on M). In order to state volume comparison theorems introduced below, we need the following concepts.

Definition 2.2. Given a continuous function $k : [0, l) \rightarrow \mathbb{R}$, we say that M has a radial Ricci curvature lower bound $(n-1)k$ at the point x if

$$\text{Ric}(v_z, v_z) \geq (n-1)k(d(x, z)), \quad \forall z \in M \setminus \text{Cut}(x) \cup \{x\},$$

where Ric is the Ricci curvature of M .

Definition 2.3. Given a continuous function $k : [0, l) \rightarrow \mathbb{R}$, we say that M has a radial sectional curvature upper bound k along any unit-speed minimizing geodesic starting from a point $x \in M$ if

$$K(v_z, V) \leq k(d(x, z)), \quad \forall z \in M \setminus (\text{Cut}(x) \cup \{x\}),$$

where $V \perp v_z$, $V \in S_z^{n-1} \subseteq T_z M$, and $K(v_z, V)$ is the sectional curvature of the plane spanned by v_z and V .

Remark 2.4. As in Subsection 2.1, in Definitions 2.2 and 2.3, $\text{Cut}(x)$ is the cut-locus of x on M , and $v_z \in S_p^{n-1} \subseteq T_z M$ is the unit tangent vector of the minimizing geodesic $\gamma_{x,z}$ emanating from x and joining x and z . Clearly, v_z is in the radial direction. In fact, the notion of having radial curvature

bound has been used by the author in [12, 22, 23] to investigate some problems like eigenvalue comparisons for the Laplace and p -Laplace operators (between the given complete manifold and its model manifold), the heat kernel comparison, etc. This notion can also be found in other literatures (see, for instance, [17, 29]). Let $t := d(x, \cdot)$, the inequality in Definition 2.2 (resp., Definition 2.3) becomes $\text{Ric}(v_z, v_z) \geq (n-1)k(t)$ (resp., $K(v_z, V) \leq k(t)$) for any $z \in M \setminus \text{Cut}(x) \cup \{x\}$. We also say that the radial Ricci (resp., sectional) curvature of M is bounded from below (resp., above) by $(n-1)k(t)$ (resp., $k(t)$) w.r.t. $x \in M$ if the above inequality is satisfied.

Define a function $\tilde{\theta}(t, \xi)$ on $M \setminus \text{Cut}(x)$ as follows

$$\tilde{\theta}(t, \xi) = \left[\frac{J(t, \xi)}{h(t)} \right]^{n-1}.$$

Then we have the following volume comparison result, which corresponds to [12, Theorem 3.3 and Corollary 3.4] (equivalently, [22, Theorem 2.6] or [23, Theorem 2.2.3 and Corollary 2.2.4]).

Theorem 2.5. *(A generalized Bishop's volume comparison theorem I) Given $\xi \in S_x^{n-1} \subseteq T_x M$, and a model space $M^* = [0, l] \times_h \mathbb{S}^{n-1}$ with the base point p^* , under the curvature assumption on the radial Ricci tensor, $\text{Ric}(v_z, v_z) \geq -(n-1)h''(t)/h(t)$ on M , for $z = \gamma_\xi(t) = \exp_x(t\xi)$ with $t < \min\{d_\xi, l\}$, the function $\tilde{\theta}$ is non-increasing in t . In particular, for all $t < \min\{d_\xi, l\}$ we have $J(t, \xi) \leq h(t)$. Furthermore, this inequality is strict for all $t \in (t_0, t_1]$, with $0 \leq t_0 < t_1 < \min\{d_\xi, l\}$, if the above curvature assumption holds with a strict inequality for t in the same interval. Besides, for $r_0 < \min\{l(x), l\}$ with $l(x)$ defined by (2.4), we have*

$$\text{vol}[B(x, r_0)] \leq \text{vol}[\tilde{B}(p^*, r_0)],$$

with equality if and only if $B(x, r_0)$ is isometric to $\tilde{B}(p^*, r_0)$.

Similarly, we have the following volume comparison conclusion, which corresponds to [12, Theorem 4.2] (equivalently, [22, Theorem 2.7] or [23, Theorem 2.3.2]).

Theorem 2.6. *(A generalized Bishop's volume comparison theorem II) Assume M has a radial sectional curvature upper bound $k(t) = -\frac{h''(t)}{h(t)}$ w.r.t. $x \in M$ for $t < \beta \leq \min\{inj_c(x), l\}$, where $inj_c(x) = \inf_\xi c_\xi$, with $\gamma_\xi(c_\xi)$ a first conjugate point along the geodesic $\gamma_\xi(t) = \exp_x(t\xi)$. Then on $(0, \beta)$*

$$\left(\frac{\sqrt{|g|}}{h^{n-1}} \right)' \geq 0, \quad \sqrt{|g|}(t) \geq h^{n-1}(t),$$

and equality occurs in the first inequality at $t_0 \in (0, \beta)$ if and only if

$$\mathcal{R} = -\frac{h''(t)}{h(t)}, \quad \mathbb{A} = h(t)I,$$

on all of $[0, t_0]$.

2.3 A volume comparison theorem for smooth metric measure spaces with weighted Ricci curvature bounded from below

As mentioned at the beginning of this section, the following volume comparison theorem proven by Wei and Wylie (cf. [31, Theorem 1.2]) is the key point to prove Theorem 1.2.

Theorem 2.7. [31] *Let $(M, g, e^{-f} dv_g)$ be n -dimensional ($n \geq 2$) complete smooth metric measure space with $\text{Ric}_f \geq (n-1)H$. Fix $x_0 \in M$. If $\partial_t f \geq -a$ along all minimal geodesic segments from x_0 then for $R \geq r > 0$ (assume $R \leq \pi/2\sqrt{H}$ if $H > 0$),*

$$\frac{\text{vol}_f[B(x_0, R)]}{\text{vol}_f[B(x_0, r)]} \leq e^{aR} \frac{\text{vol}_H^n(R)}{\text{vol}_H^n(r)},$$

where $\text{vol}_H^n(\cdot)$ is the volume of the geodesic ball with the prescribed radius in the space n -form with constant sectional curvature H , and, as before, $\text{vol}_f(\cdot)$ denotes the weighted (or f -)volume of the given geodesic ball on M . Moreover, equality in the above inequality holds if and only if the radial sectional curvatures are equal to H and $\partial_t f \equiv -a$. In particular, if $\partial_t f \geq 0$ and $\text{Ric} \geq 0$, then M has f -volume growth of degree at most n .

Therefore, given a complete and *non-compact* smooth metric measure n -space $(M, g, e^{-f} dv_g)$, if $\partial_t f \geq 0$ (along all minimal geodesic segments from x_0) and $\text{Ric}_f \geq 0$, then by Theorem 2.7 we have

$$\frac{\text{vol}_f[B(x_0, R)]}{\text{vol}_f[B(x_0, r)]} \leq e^{0 \cdot R} \cdot \frac{V_0(R)}{V_0(r)} = \frac{V_0(R)}{V_0(r)},$$

with, as before, $V_0(\cdot)$ denotes the volume of the ball with the prescribed radius in \mathbb{R}^n , which is equivalent with

$$\frac{\text{vol}_f[B(x_0, R)]}{V_0(R)} \leq \frac{\text{vol}_f[B(x_0, r)]}{V_0(r)} \quad (2.9)$$

for $R \geq r > 0$. Letting $r \rightarrow 0$ on the right hand side of the above inequality, and together with (2.2), (2.3) and (2.6), we can get

$$\begin{aligned} \frac{\text{vol}_f[B(x_0, R)]}{V_0(R)} &\leq \lim_{r \rightarrow 0} \frac{\int_{\mathbb{S}^{n-1}} \left(\int_0^{\min\{R, d_\xi\}} J^{n-1}(t, \xi) \cdot e^{-f} dt \right) d\sigma}{\int_{\mathbb{S}^{n-1}} \int_0^R t^{n-1} dt d\sigma} \\ &= \frac{J'(0, \xi) \cdot e^{-f(x_0)}}{1} = e^{-f(x_0)} \end{aligned}$$

by applying L'Hôpital's rule n -times. Hence, if $\partial_t f \geq 0$ and $\text{Ric}_f \geq 0$, we have

$$\text{vol}_f[B(x_0, R)] \leq e^{-f(x_0)} \cdot V_0(R) \quad (2.10)$$

for $R > 0$.

3 Proofs of main results

Now, by using the facts listed in Section 2 and a similar method to that of [8, Theorem 1.1], we can prove Theorem 1.2 as follows.

Proof of Theorem 1.2. Let

$$y = 2a - bp + 2, \quad z = \frac{(n - 2a - 2)p}{2a - bp + 2} = \frac{2p}{p - 2}, \quad (3.1)$$

where $n \geq 3$ and a, b, p are constants determined by (1.4). Since $t = t(\cdot) := d(x_0, \cdot)$ is a Lipschitz continuous function from M to \mathbb{R} , then for any $\lambda > 0$, we can define a function $F(\lambda)$ as follows

$$F(\lambda) = \frac{p-2}{p+2} \int_M \frac{e^{-f} dv_g}{t^{bp} \cdot (\lambda + t^y)^{z-1}}. \quad (3.2)$$

by applying the Fubini theorem (cf. [28]) to (3.2), we have

$$F(\lambda) = \frac{p-2}{p+2} \int_0^{+\infty} \text{vol}_f \left[x : \frac{1}{t^{bp} \cdot (\lambda + t^y)^{z-1}} > s \right] ds. \quad (3.3)$$

Since $\partial_t f \geq 0$ (along all minimal geodesic segments from x_0) and $\text{Ric}_f \geq 0$, we have (2.10) by applying Theorem 2.7. By making variable change

$$s = \frac{1}{\rho^{bp} \cdot (\lambda + \rho^y)^{z-1}}$$

in (3.3) and together with (2.10), we can obtain

$$\begin{aligned} F(\lambda) &= \frac{p-2}{p+2} \int_0^{+\infty} \text{vol}_f [x : t(x) < \rho] \frac{[bp\lambda + (bp + (z-1)y)\rho^y]}{\rho^{bp+1}(\lambda + \rho^y)^z} d\rho \\ &= \frac{p-2}{p+2} \int_0^{+\infty} \text{vol}_f [B(x_0, \rho)] \frac{[bp\lambda + (bp + (z-1)y)\rho^y]}{\rho^{bp+1}(\lambda + \rho^y)^z} d\rho. \end{aligned} \quad (3.4)$$

On the other hand, by (1.4) and (3.1), we can get

$$n - bp - 1 > -1, \quad n - bp - 1 + y(1 - z) < -1.$$

Substituting the above fact into (3.4), it is easy to know that $0 \leq F(\lambda) < +\infty$ for any $\lambda > 0$. Besides, we also have

$$F'(\lambda) = - \int_M \frac{e^{-f} dv_g}{t^{bp} \cdot (\lambda + t^y)^z}.$$

Therefore, from the above argument, it follows that F defined by (3.2) is differentiable. Since for every $\lambda > 0$, $(\lambda + t^y)^{-\frac{z}{p}}$ is a continuous function and tends to zero as $t \rightarrow +\infty$, which implies that

there exists at least a sequence of functions $\{g_n(t)\}$ in $C_0^\infty(M)$ such that $g_n(t) \rightarrow (\lambda + t^y)^{-\frac{z}{p}}$ as $n \rightarrow +\infty$. By the assumption (1.5) and an approximation procedure for the function $(\lambda + t^y)^{-\frac{z}{p}}$, we have

$$\begin{aligned} \left[\int_M \frac{e^{-f} dv_g}{t^{bp} \cdot (\lambda + t^y)^z} \right]^{\frac{2}{p}} &\leq \left(\frac{yzC_2}{p} \right)^2 \int_M \frac{e^{-f} dv_g}{t^{2(1+a-y)} \cdot (\lambda + t^y)^{2+\frac{2z}{p}}} \\ &= \left(\frac{yzC_2}{p} \right)^2 \int_M \frac{e^{-f} dv_g}{t^{bp-y} \cdot (\lambda + t^y)^z}. \end{aligned}$$

Let $\ell := \left(\frac{p}{yzC_2} \right)^2$. Then the above inequality can be rewritten as follows

$$\ell [-F'(\lambda)]^{\frac{2}{p}} \leq \lambda F'(\lambda) + \frac{p+2}{p-2} F(\lambda). \quad (3.5)$$

Consider the function $G : (0, +\infty) \rightarrow \mathbb{R}$ defined by

$$G(\lambda) := \frac{p-2}{p+2} \cdot e^{-f(x_0)} \cdot \int_{\mathbb{R}^n} \frac{dv_{\mathbb{R}^n}}{|x|^{bp} \cdot (\lambda + |x|^y)^{z-1}},$$

where, as before, $|x|$ denotes the length of the vector $x \in \mathbb{R}^n$. Since, as mentioned in Section 1, when $C = K_{a,b}$, the extremal functions in the Caffarelli-Kohn-Nirenberg inequality (1.2) are of the form $u_\lambda := (\lambda + |x|^y)^{-\frac{z}{p}}$, $\lambda > 0$, we have

$$\begin{aligned} [-G'(\lambda)]^{\frac{2}{p}} &= \left(e^{-f(x_0)} \cdot \int_{\mathbb{R}^n} \frac{dv_{\mathbb{R}^n}}{|x|^{bp} \cdot (\lambda + |x|^y)^z} \right)^{\frac{2}{p}} \\ &= \left(\frac{yzK_{a,b}}{p} \right)^2 \left(e^{-f(x_0)} \right)^{\frac{2}{p}} \int_{\mathbb{R}^n} \frac{dv_{\mathbb{R}^n}}{|x|^{2(1+a-y)} \cdot (\lambda + |x|^y)^{2+\frac{2z}{p}}} \\ &= \left(\frac{yzK_{a,b}}{p} \right)^2 \cdot \left[\lambda G'(\lambda) + \frac{p+2}{p-2} G(\lambda) \right]. \end{aligned} \quad (3.6)$$

Together with the fact $G(\lambda) = G(1)\lambda^{-\frac{2}{p-2}}$, it follows that

$$\begin{aligned} G(1) &= \frac{p-2}{p+2} \cdot e^{-f(x_0)} \cdot \int_{\mathbb{R}^n} \frac{dv_{\mathbb{R}^n}}{|x|^{bp} \cdot (1 + |x|^y)^{z-1}} \\ &= 2^{\frac{2}{p-2}} (p-2) [(n-2a-2)K_{a,b}]^{-\frac{2p}{p-2}}. \end{aligned} \quad (3.7)$$

Define function $H(\lambda)$, $\lambda > 0$, given by

$$H(\lambda) := A\lambda^{-\frac{2}{p-2}}, \quad (3.8)$$

where A satisfies

$$\begin{aligned}
 A &= 2^{\frac{2}{p-2}}(p-2) \left(\frac{\ell}{p} \right)^{\frac{p}{p-2}} \\
 &= \left(\frac{K_{a,b}}{C_2} \right)^{\frac{2p}{p-2}} \cdot 2^{\frac{2}{p-2}}(p-2) [(n-2a-2)K_{a,b}]^{-\frac{2p}{p-2}} \\
 &= \left(\frac{K_{a,b}}{C_2} \right)^{\frac{n}{1+a-b}} \cdot \frac{p-2}{p+2} \cdot e^{-f(x_0)} \cdot \int_{\mathbb{R}^n} \frac{dv_{\mathbb{R}^n}}{|x|^{bp} \cdot (1+|x|^y)^{z-1}} \\
 &= \left(\frac{K_{a,b}}{C_2} \right)^{\frac{n}{1+a-b}} \cdot G(1).
 \end{aligned}$$

Clearly, by (3.7) and (3.8), we know that

$$H(\lambda) = \left(\frac{K_{a,b}}{C_2} \right)^{\frac{n}{1+a-b}} G(\lambda). \quad (3.9)$$

Combining (3.6) and (3.9), one can easily check that $H(\lambda)$ satisfies the following differential equation

$$\ell [-H'(\lambda)]^{\frac{2}{p}} = \lambda H'(\lambda) + \frac{p+2}{p-2} H(\lambda). \quad (3.10)$$

By L'Hôpital's rule, we have

$$\lim_{\rho \rightarrow 0} \frac{\text{vol}_f[B(x_0, \rho)]}{V_0(\rho)} = \lim_{\rho \rightarrow 0} \frac{\int_{S_{x_0}^{n-1}} \left(\int_0^{\min\{\rho, d_\xi\}} J^{n-1}(t, \xi) \cdot e^{-f} dt \right) d\sigma}{w_n \int_0^\rho t^{n-1} dt} = e^{-f(x_0)}.$$

So, for a fixed small $\varepsilon > 0$, there exists a number $\eta > 0$ such that $\text{vol}_f[B(x_0, \rho)] \geq (1 - \varepsilon)e^{-f(x_0)} \cdot V_0(\rho)$, $\forall \rho \leq \eta$. Together this fact with (3.4), we can get

$$\begin{aligned}
 F(\lambda) &\geq \frac{p-2}{p+2} (1 - \varepsilon) \cdot e^{-f(x_0)} \cdot \int_0^\eta V_0(\rho) \frac{[bp\lambda + (bp + (z-1)y)\rho^y]}{\rho^{bp+1}(\lambda + \rho^y)^z} d\rho \\
 &= \frac{p-2}{p+2} (1 - \varepsilon) \lambda^{\frac{n+bp}{y} + 1 - z} \cdot e^{-f(x_0)} \cdot \int_0^{\eta/\lambda^{\frac{1}{y}}} V_0(s) \frac{[bp + (bp + (z-1)y)s^y]}{s^{bp+1}(1 + s^y)^z} ds \\
 &= \frac{p-2}{p+2} (1 - \varepsilon) \lambda^{-\frac{2}{p-2}} \cdot e^{-f(x_0)} \cdot \int_0^{\eta/\lambda^{\frac{1}{y}}} V_0(s) \frac{[bp + (bp + (z-1)y)s^y]}{s^{bp+1}(1 + s^y)^z} ds.
 \end{aligned}$$

On the other hand, by a direct computation, we have

$$G(\lambda) = \frac{p-2}{p+2} \lambda^{-\frac{2}{p-2}} \cdot e^{-f(x_0)} \cdot \int_0^{+\infty} V_0(s) \frac{[bp + (bp + (z-1)y)s^y]}{s^{bp+1}(1 + s^y)^z} ds. \quad (3.11)$$

Therefore, it is easy to observe that

$$\liminf_{\lambda \rightarrow 0} \frac{F(\lambda)}{G(\lambda)} \geq 1 - \varepsilon,$$

and from which, one can obtain

$$\liminf_{\lambda \rightarrow 0} \frac{F(\lambda)}{G(\lambda)} \geq 1 \quad (3.12)$$

by letting $\varepsilon \rightarrow 0$.

Now, we divide into two cases to prove the assertion of Theorem 1.2 as follows.

Case (I): $C_2 > K_{a,b}$.

In this case, by (3.9) and (3.12), it follows that

$$\liminf_{\lambda \rightarrow 0} \frac{F(\lambda)}{H(\lambda)} = \left(\frac{C_2}{K_{a,b}} \right)^{\frac{n}{1+a-b}} \liminf_{\lambda \rightarrow 0} \frac{F(\lambda)}{G(\lambda)} \geq \left(\frac{C_2}{K_{a,b}} \right)^{\frac{n}{1+a-b}} > 1. \quad (3.13)$$

On the other hand, we **claim** that if there exists some $\lambda_0 > 0$ such that $F(\lambda_0) < H(\lambda_0)$, then we have $F(\lambda) < H(\lambda)$, $\forall \lambda \in (0, \lambda_0]$. We will prove this by contradiction. Assume that there exists some $\tilde{\lambda} \in (0, \lambda_0)$ such that $F(\tilde{\lambda}) \geq H(\tilde{\lambda})$. Then we can set

$$\lambda_1 := \sup \left\{ \tilde{\lambda} < \lambda_0 \mid F(\tilde{\lambda}) \geq H(\tilde{\lambda}) \right\}.$$

So, we have $0 < F(\lambda) \leq H(\lambda)$ for any $\lambda_1 \leq \lambda \leq \lambda_0$. For each $\lambda > 0$, define a function $\phi_\lambda : [0, +\infty) \rightarrow \mathbb{R}$ given by

$$\phi_\lambda(m) = \ell \cdot m^{\frac{2}{p}} + \lambda \cdot m.$$

Clearly, $\phi_\lambda(m)$ is increasing on $[0, +\infty)$. Therefore, together with (3.5) and (3.10), it is not difficult to get

$$\begin{aligned} F'(\lambda) - H'(\lambda) &\geq -\phi_\lambda^{-1} \left(\frac{p+2}{p-2} F(\lambda) \right) + \phi_\lambda^{-1} \left(\frac{p+2}{p-2} H(\lambda) \right) \\ &= \phi_\lambda^{-1} \left(\frac{p+2}{p-2} (H(\lambda) - F(\lambda)) \right) \geq \phi_\lambda^{-1}(0) = 0 \end{aligned}$$

for any $\lambda_1 \leq \lambda \leq \lambda_0$. So, $(F - H)'(\lambda) \leq 0$ on $[\lambda_1, \lambda_0]$. Consequently, we can obtain

$$0 \geq (F - H)(\lambda_1) \leq (F - H)(\lambda_0) < 0,$$

which is clearly a contradiction. Thus the **claim** above is true.

By (3.13) and the above **claim**, we have

$$F(\lambda) \geq H(\lambda), \quad \forall \lambda > 0.$$

Consequently, together with (3.4), (3.9) and (3.11), we get that for any $\lambda > 0$, the following inequality

$$\int_0^{+\infty} \left[\text{vol}_f[B(x_0, \rho)] - \left(\frac{K_{a,b}}{C_2} \right)^{\frac{n}{1+a-b}} \cdot e^{-f(x_0)} \cdot V_0(\rho) \right] \cdot \frac{[bp\lambda + (bp + (z-1)y)\rho^y]}{\rho^{bp+1}(\lambda + \rho^y)^z} d\rho \geq 0 \quad (3.14)$$

hods. Let $b = \left(\frac{K_{a,b}}{C_2}\right)^{\frac{n}{1+a-b}}$. Clearly, $0 < b < 1$. By Theorem 2.7, when $\partial_f \geq 0$ (along all minimal geodesic segments from x_0) and $\text{Ric}_f \geq 0$, we have (2.9) holds for $R \geq r > 0$ and (2.10) holds for $R > 0$, which implies that the volume ratio $\text{vol}_f[B(x_0, \rho)]/V_0(\rho)$ is non-increasing for $t \in (0, +\infty)$. Assume now that $\lim_{\rho \rightarrow +\infty} \frac{\text{vol}_f[B(x_0, \rho)]}{e^{-f(x_0)} \cdot V_0(\rho)} = b_0$. Clearly, $b_0 \leq 1$. Now, *in order to get the conclusion of Theorem 1.2 in the case $C_2 > K_{a,b}$, it is sufficient to show that $b_0 \geq b$* . We will prove this fact by contradiction. Assume that $b_0 = b - \varepsilon_0$ for some $\varepsilon_0 > 0$. Then there exists some $N_0 > 0$ such that

$$\frac{\text{vol}_f[B(x_0, \rho)]}{e^{-f(x_0)} \cdot V_0(\rho)} \leq b - \frac{\varepsilon_0}{2}, \quad \forall t \geq N_0.$$

Substituting the above inequality into (3.14), and together with (2.10), we have

$$\begin{aligned} 0 &\leq \int_0^{N_0} \frac{\text{vol}_f[B(x_0, \rho)]}{e^{-f(x_0)} \cdot V_0(\rho)} \cdot \frac{\rho^n [bp\lambda + (bp + (z-1)y)\rho^y]}{\rho^{bp+1}(\lambda + \rho^y)^z} d\rho + \\ &\quad \int_{N_0}^{+\infty} \left(b - \frac{\varepsilon_0}{2}\right) \cdot \frac{\rho^n [bp\lambda + (bp + (z-1)y)\rho^y]}{\rho^{bp+1}(\lambda + \rho^y)^z} d\rho - b \int_0^{+\infty} \frac{\rho^n [bp\lambda + (bp + (z-1)y)\rho^y]}{\rho^{bp+1}(\lambda + \rho^y)^z} d\rho \\ &\leq \int_0^{N_0} \frac{\rho^n [bp\lambda + (bp + (z-1)y)\rho^y]}{\rho^{bp+1}(\lambda + \rho^y)^z} d\rho + \int_{N_0}^{+\infty} \left(b - \frac{\varepsilon_0}{2}\right) \cdot \frac{\rho^n [bp\lambda + (bp + (z-1)y)\rho^y]}{\rho^{bp+1}(\lambda + \rho^y)^z} d\rho - \\ &\quad b \int_0^{+\infty} \frac{\rho^n [bp\lambda + (bp + (z-1)y)\rho^y]}{\rho^{bp+1}(\lambda + \rho^y)^z} d\rho \\ &= \int_0^{N_0} \left(1 - b + \frac{\varepsilon_0}{2}\right) \cdot \frac{\rho^n [bp\lambda + (bp + (z-1)y)\rho^y]}{\rho^{bp+1}(\lambda + \rho^y)^z} d\rho - \\ &\quad \frac{\varepsilon_0}{2} \int_0^{+\infty} \frac{\rho^n [bp\lambda + (bp + (z-1)y)\rho^y]}{\rho^{bp+1}(\lambda + \rho^y)^z} d\rho \\ &= \int_0^{N_0} \left(1 - b + \frac{\varepsilon_0}{2}\right) \cdot \frac{\rho^n [bp\lambda + (bp + (z-1)y)\rho^y]}{\rho^{bp+1}(\lambda + \rho^y)^z} d\rho - \frac{n\varepsilon_0}{2w_n} \cdot \frac{p+2}{p-2} \cdot e^{f(x_0)} \cdot G(\lambda) \\ &\leq \left(1 - b + \frac{\varepsilon_0}{2}\right) \lambda^{-z} \int_0^{N_0} \left[bp\lambda \rho^{n-bp-1} + (bp + (z-1)y)\rho^{n+y-bp-1}\right] d\rho - \\ &\quad \frac{n\varepsilon_0}{2w_n} \cdot \frac{p+2}{p-2} \cdot e^{f(x_0)} \cdot \lambda^{-\frac{2}{p-2}} \cdot G(1) \\ &= \left(1 - b + \frac{\varepsilon_0}{2}\right) \lambda^{-z} \left[\frac{\lambda bp N_0^{n-bp}}{n-bp} + \frac{(bp + (z-1)y) N_0^{n+y-bp}}{n+y-bp} \right] - \\ &\quad \frac{n\varepsilon_0(p+2)G(1)}{2w_n(p-2)} \cdot e^{f(x_0)} \cdot \lambda^{-\frac{2}{p-2}} \end{aligned}$$

for every $\lambda > 0$. Hence, for any $\lambda > 0$, we have

$$0 < \frac{n\varepsilon_0(p+2)G(1)}{2w_n(p-2)} \cdot e^{f(x_0)} \leq \left(1 - b + \frac{\varepsilon_0}{2}\right) \cdot \lambda^{\frac{2}{p-2}-z} \cdot \left[\frac{\lambda bp N_0^{n-bp}}{n-bp} + \frac{(bp+(z-1)y)N_0^{n+y-bp}}{n+y-bp} \right].$$

Since $\frac{2}{p-2} - z + 1 < 0$, one can get a contradiction by letting $\lambda \rightarrow +\infty$ in the above inequality. So, this completes the proof of the conclusion of Theorem 1.2 for the case $C_2 > K_{a,b}$.

Case (2): $C_2 = K_{a,b}$.

In this case, by the assumption (1.5), we have for any fixed $\gamma > 0$ that

$$\left(\int_M t^{-bp} |u|^p \cdot e^{-f} dv_g \right)^{\frac{1}{p}} \leq (K_{a,b} + \gamma) \left(\int_M t^{-2a} |\nabla u|^2 \cdot e^{-f} dv_g \right)^{\frac{1}{2}}.$$

Then, by the same argument to *Case (1)*, we can obtain

$$\text{vol}_f[B(x_0, r)] \geq \left(\frac{K_{a,b}}{K_{a,b} + \gamma} \right)^{\frac{n}{1+a-b}} \cdot e^{-f(x_0)} \cdot V_0(r), \quad \forall r > 0,$$

which, by letting $\gamma \rightarrow 0$, implies

$$\text{vol}_f[B(x_0, r)] \geq e^{-f(x_0)} \cdot V_0(r), \quad \forall r > 0.$$

This completes the proof of the conclusion of Theorem 1.2 for the case $C_2 = K_{a,b}$. □

Now, we give the proof of Corollary 1.4 as follows.

Proof of Corollary 1.4. By Theorem 1.2 directly, we have

$$\text{vol}_f[B(x_0, r)] \geq e^{-f(x_0)} \cdot V_0(r), \quad \forall r > 0.$$

However, from (2.10) which is obtained by Theorem 2.7, we have

$$\text{vol}_f[B(x_0, r)] \leq e^{-f(x_0)} \cdot V_0(r), \quad \forall r > 0.$$

Therefore, we have

$$\text{vol}_f[B(x_0, r)] = e^{-f(x_0)} \cdot V_0(r), \quad \forall r > 0,$$

which, together with Theorem 2.7, implies that the *radial* sectional curvatures are equal to 0 and $\partial_t f \equiv 0$. So, we know that f is a constant function with respect to t , i.e., $f \equiv f(x_0)$. Besides, since the radial sectional curvatures are equal to 0, by applying Theorems 2.5 and 2.6 *simultaneously*, we have

$$\text{vol}[B(x_0, r)] = V_0(r), \quad \forall r > 0,$$

and $B(x_0, r)$ is isometric to a ball of radius r in \mathbb{R}^n for any $r > 0$, which is equivalent to say that (M, g) is isometric to $(\mathbb{R}^n, g_{\mathbb{R}^n})$. This completes the proof of Corollary 1.4. □

Acknowledgments

This work was supported by the starting-up research fund (Grant No. HIT(WH)201320) supplied by Harbin Institute of Technology (Weihai), the project (Grant No. HIT.NSRIF.2015101) supported by Natural Scientific Research Innovation Foundation in Harbin Institute of Technology, and the NSF of China (Grant No. 11401131).

References

- [1] T. Aubin, Problèmes isopérimétriques et espaces de Sobolev, *J. Differential Geom.* **11** (1976) 573–598.
- [2] T. Aubin, Nonlinear Analysis on Manifolds, Monge-Ampère Equations, *Springer*, Berlin, 1982.
- [3] T. Aubin, Some Nonlinear Problems in Riemannian Geometry, *Springer*, Berlin, 1998.
- [4] D. Bakry and M. Émery, Hypercontractivité de semi-groupes de diffusion, *C. R. Acad. Sci. Paris Sér. I Math.* **299** (1984) 775–778.
- [5] D. Bakry and M. Émery, Diffusions hypercontractives, Séminaire de probabilités, XIX, 1983/84, Lecture Notes in Math., vol. 1123, *Springer*, Berlin, 1985, pp. 177–206.
- [6] L.-A. Caffarelli, R. Kohn and L. Nirenberg, First order interpolation inequalities with weights, *Compositio Math.* **53** (1984) 259–275.
- [7] M.-P. do Carmo and C.-Y. Xia, Ricci curvature and the topology of open manifolds, *Math. Ann.* **316** (2000) 319–400.
- [8] M.-P. do Carmo and C.-Y. Xia, Complete manifolds with non-negative Ricci curvature and the Caffarelli-Kohn-Nirenberg inequalities, *Compositio Math.* **140** (2004) 818–826.
- [9] F. Catrina and Z.-Q. Wang, On the Caffarelli-Kohn-Nirenberg inequalities: sharp constants, existence (and non-existence), and symmetry of extremal functions, *Comm. Pure Appl. Math.* **54** (2001) 229–258.
- [10] I. Chavel, Eigenvalues in Riemannian geometry, *Academic Press*, New York, 1984.
- [11] K.-S. Chou and C.-W. Chu, On the best constants for a weighted Sobolev-Hardy inequality, *J. London Math. Soc.* **48** (1993) 137–151.
- [12] P. Freitas, J. Mao and I. Salavessa, Spherical symmetrization and the first eigenvalue of geodesic disks on manifolds, *Calc. Var. Partial Differential Equations* (2013) DOI: 10.1007/s00526-013-0692-7.
- [13] A. Gray, Tubes, *Addison-Wesley*, New York, 1990.
- [14] G.-H. Hardy, J.-E. Littlewood and G. Pélya, Inequalities, Second Edition, *Cambridge University Press*, Cambridge, 1952.

- [15] E. Hebey, Sobolev spaces on Riemannian manifolds, Lecture notes in Mathematics, vol. 1635, *Springer*, Berlin, 1996.
- [16] N.-N. Katz and K. Kondo, Generalized space forms, *Trans. Amer. Math. Soc.* **354** (2002) 2279–2284.
- [17] K. Kondo and M. Tanaka, Total curvatures of model surfaces control topology of complete open manifolds with radial curvature bounded below: I, *Math. Ann.* **351** (2011) 251–266.
- [18] M. Ledoux, On manifolds with non-negative Ricci curvature and Sobolev inequalities, *Comm. Anal. Geom.* **7** (1999) 347–353.
- [19] E.-H. Lieb, Sharp constants in the Hardy-Littlewood-Sobolev and related inequalities, *Ann. of Math.* **118** (1983) 349–374.
- [20] J. Lott, Some geometric properties of the Bakry-Émery-Ricci tensor, *Comment. Math. Helv.* **78** (2003) 865–883.
- [21] J. Mao, Open manifold with nonnegative Ricci curvature and collapsing volume, *Kyushu J. Math.* **66** (2) (2012) 509–516.
- [22] J. Mao, Eigenvalue inequalities for the p -Laplacian on a Riemannian manifold and estimates for the heat kernel, *J. Math. Pures Appl.* **101** (3) (2014) 372–393.
- [23] J. Mao, Eigenvalue estimation and some results on finite topological type, Ph.D. thesis, IST-UTL, 2013.
- [24] J. Mao, The Gagliardo-Nirenberg inequalities and manifolds with nonnegative weighted Ricci curvature, (2014) submitted.
- [25] J. Mao, Functional inequalities and manifolds with nonnegative weighted Ricci curvature, (2014) submitted.
- [26] G. Perelman, The entropy formula for the Ricci flow and its geometric applications, available online at arXiv: math.DG/0211159.
- [27] P. Petersen, Comparison geometry problem list. Riemannian Geometry (Waterloo, ON, 1993) (Fields Inst. Monogr., 4). *American Mathematical Society*, Providence, RI, 1996, pp. 87–115.
- [28] R. Schoen and S.-T. Yau, Lectures on Differential Geometry, *International Press*, Cambridge, MA, 1994.
- [29] K. Shiohama and M. Tanaka, Compactification and maximal diameter theorem for noncompact manifolds with radial curvature bounded below, *Math. Z.* **241** (2002) 341–351.
- [30] G. Talenti, Best constant in Sobolev inequality, *Ann. Mat. Pura Appl.* **110** (1976) 353–372.
- [31] G.-F. Wei and W. Wylie, Comparison geometry for the Bakry-Émery Ricci tensor, *J. Differential Geom.* **83** (2009) 377–405.
- [32] C.-Y. Xia, Complete manifolds with non-negative Ricci curvature and almost best constant, *Illinois J. Math.* **45** (2001) 1253–1259.